Section 2.3 – Arithmetic Sequence and Series

This booklet belongs to:______________________________Block: ______

A Sequence

- A sequence is simply a list of numbers
- Each number in the list is called a **Term**
- They are listed: first term, second term, third term, and so on…
- Sequences can be finite (they end) or infinite (they don't end)

Unlike graphing equations and using function notation of $x \text{ and } y \text{ values}$ sequences use script notation.

Example:

Term 1 is known as: a_1 or t_1

Term 2 is known as: a_2 or t_2

Term 3 is known as: a_3 or t_3

… … …

The n^{th} is known as: a_n or t_n

- Depending on the resource, some people use a other use t when denoting the first term.
- n refers to whichever number you want to input

Sequence

A **finite sequence** is a function for which the $\boldsymbol{domain}$ $(\boldsymbol{x-values})$ is a subset of the natural numbers: $\{1, 2, 3, \dots, n\}$ for some finite number n

An **infinite sequence** is function for which the $\boldsymbol{domain}$ $(\boldsymbol{x-values})$ is the set of natural numbers: $\{1, 2, 3, \dots\}$

Example 1: Write the first four terms of the sequence

a) $a_n = \frac{n+1}{n}$ b) $b_n = 2n - 3$ c) $t_n = 2^n$

Solution 1:

a) $a_1 = \frac{1+1}{1} = 2,$ $a_2 = \frac{2+1}{2} = \frac{3}{2},$ $a_3 = \frac{3+1}{3} = \frac{4}{3},$ $a_4 = \frac{4+1}{4} = \frac{5}{4}$

b) $b_1 = 2(1) - 3 = -1,$ $b_2 = (2)(2) - 3 = 1,$ $b_3 = (2)(3) - 3 = 3,$ $b_4 = (2)(4) - 3 = 5$

c) $t_1 = 2^1 = 2,$ $t_2 = 2^2 = 4,$ $t_3 = 2^3 = 8,$ $t_4 = 2^4 = 16$

Arithmetic Sequence

- When we have a **sequence in which the successive terms** have a **common difference**, the sequence is called and **arithmetic sequence**
- For example, the sequence, $3, 7, 11, 15, \dots$ has a **common difference** of 4. Every next term is achieved **by adding 4** to the term previous.
- The common difference, $\boldsymbol{d}$, of this sequence is 4.

If we look at the pattern we may see something helpful…

1st term: $a_1 = a_1$

2nd term: $\boldsymbol{a_2 = a_1 + d}$

3rd term: $\boldsymbol{a_3 = a_2} + d = (\boldsymbol{a_1 + d}) + d = \boldsymbol{a_1 + 2d}$

4th term: $\boldsymbol{a_4 = a_3} + d = (\boldsymbol{a_1 + 2d}) + d = \boldsymbol{a_1 + 3d}$

> We are able to express every term with respect to the first term and the common difference!

- From this pattern we are able to generate the general equation of an Arithmetic Sequence

The $\boldsymbol{n^{th}}$ term of an Arithmetic Sequence

- For an arithmetic sequence $\{\boldsymbol{t_n}\}$ whose first term is $\boldsymbol{a}$, with a common difference $\boldsymbol{d}$:

$$\boldsymbol{t_n = a + (n-1)d} \qquad \text{for any integer } n \geq 1$$

Example 2: For each arithmetic sequence, identity the common difference.

a) $3, 5, 7, 9, \ldots$
b) $11, 8, 5, 2, \ldots$

Solution 2:

a) $5 - 3 = 2$, $7 - 5 = 2$, $9 - 7 = 2$, Therefore $d = 2$
b) $8 - 11 = -3$, $5 - 8 = -3$, $2 - 5 = -3$, Therefore $d = -3$

Example 3: Determine if the sequence $\{t_n\} = \{3 - 2n\}$ is arithmetic

Solution 3:

$t_1 = 3 - 2(1) = 1$

$t_2 = 3 - 2(2) = -1$

$t_3 = 3 - 2(3) = -3$

$1, -1, -3, \ldots$ has a common difference of -2

So, the sequence is arithmetic!

Example 4: Find the 12th term of the arithmetic sequence $2, 5, 8, \ldots$

Solution 4:

$a = 2$ $d = 3$

$t_n = a + (n - 1)d$

$t_{12} = 2 + (12 - 1)3 \rightarrow 35$

Example 5: Which term in the arithmetic sequence $4, 7, 10, \ldots$ has a value of 439?

Solution 5:

$d = 7 - 4 = 3$

$t_n = a + (n - 1)d$

$439 = 4 + (n - 1)3$

$435 = (n - 1)3 \rightarrow 145 = n - 1$

$n = 146$ The 146th term is 439.

Example 6: The 7th term of an arithmetic sequence is 78, and the 18th term is 45. Find the 1st term.

Solution 6:

There are $18 - 7 = \mathbf{11}$ **terms** between 45 and 78. And the difference between them is $\mathbf{45 - 78 = -33}$

So, $11d = -33 \quad \rightarrow \quad d = -3$

This gets us the common difference by using the gap in all the terms we missed.

So,

$t_n = a + (n-1)d$

$t_7 = a + (7-1)(-3)$

$78 = a + (-18)$

$78 + 18 = a = 96$

Can use the 7th term or 18th

Example 7: Find x so that $3x + 2, 2x - 3, and\ 2 - 4x$ are consecutive terms of an arithmetic sequence

Solution 7:

Since they are consecutive,

$(2x - 3) - (3x + 2) = d$ and $(2 - 4x) - (2x - 3) = d$

So, since they both equal d, we can set them equal to each other.

$$(2x - 3) - (3x + 2) = (2 - 4x) - (2x - 3)$$

$$2x - 3 - 3x - 2 = 2 - 4x - 2x + 3$$

$$-x - 5 = -6x + 5 \quad \rightarrow \quad 5x = 10 \quad \rightarrow \quad x = 2$$

Once you have x, you can then work backwards to determine d and the first three terms.

$(2x - 3) - (3x + 2) = d \quad \rightarrow \quad (2(2) - 3) - (3(2) + 2) = d \quad \rightarrow \quad 1 - 8 = d \quad \rightarrow \quad \boldsymbol{d = -7}$

$$3x + 2 \quad \rightarrow \quad 3(2) + 2 = 8$$
$$2x - 3 \quad \rightarrow \quad 2(2) - 3 = 1$$
$$2 - 4x \quad \rightarrow \quad 2 - 4(2) = -6$$

First 3 terms are: $8, 1, -6$

Arithmetic Series

- An **arithmetic series** is when we take our given **sequence** and we **add it all together** (sum)
- We have **finite and infinite sums** just like we have for sequences, but we're only going to look at **finite series**
- Here's the formula:

Sum of an Arithmetic Series

- The sum of the first n terms of an arithmetic series is given by:

$$S_n = \frac{n}{2}(a + l) \qquad or \qquad S_n = \frac{n}{2}(2a + (n-1)d)$$

Where $a = the\ first\ term,\quad l = the\ last\ term,$ and $d = the\ common\ difference$

- We can interchange the two equations, depending on what information is given to us
- Then it really just becomes plug by numbers

Example 8: Find the sum of the positive integers from $1\ to\ 50$ inclusive.

Solution 8:

$$a = 1, \quad l = 50, \quad d = 1$$

$$S_n = \frac{n}{2}(a + l)$$

Since we have $\boldsymbol{a\ and\ l}$ we know we can use this one.

$$S_{50} = \frac{50}{2}(1 + 50) \quad \rightarrow \quad 25(51) \quad \rightarrow \quad 1275$$

Example 9: Find the sum of the first 25 terms of the series $11 + 15 + 19 + \cdots$

Solution 9: The series is arithmetic (has a common difference) with $a = 11, d = 4, and\ n = 25$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

Since we **don't know $\boldsymbol{l}$** we know we have to use this one.

$$S_{25} = \frac{25}{2}(2(11) + (25-1)4) \quad \rightarrow \quad 12.5(22 + 96) \quad \rightarrow \quad 1475$$

Example 10: Find the sum of the series $7 + 10 + 13 + \cdots + 100$.

Solution 10: $a = 7, l = 100, d = 3$ but We **don't know n**, so we solve for that first

To **find n** we use the **formula from Section 2.2**

$$t_n = a + (n-1)d$$

Since we want to know **how many terms** there are and **100 is the last term**, if we solve for that we'll get n.

$$100 = 7 + (n-1)(3)$$

$$\rightarrow 100 = 7 + 3n - 3$$

$$\rightarrow 100 = 4 + 3n$$

$$\rightarrow 96 = 3n$$

$$n = 32$$

Now we can solve for the sum since we know n

$$S_n = \frac{n}{2}(a + l)$$

$$S_{32} = \frac{32}{2}(7 + 100)$$

$$S_{32} = 16(107)$$

$$S_{32} = 1712$$

Example 11: Find the sum of the $5 + 9 + 13 + \cdots + 137$

Solution 11: $a = 5, l = 137, d = 4$ but We don't know n, so we solve for that first

To **find n** we use the **formula from Section 2.2**

$$t_n = a + (n-1)d$$

Since we want to **know how many terms** there are and **137 is the last term**, if we solve for that we'll get n.

$$137 = 5 + (n-1)(4)$$

$$\rightarrow 137 = 5 + 4n - 4$$

$$\rightarrow 137 = 1 + 4n$$

$$\rightarrow 136 = 4n$$

$$n = 34$$

Now we can solve for the sum since we know n

$$S_n = \frac{n}{2}(a + l)$$

$$S_{34} = \frac{34}{2}(5 + 137)$$

$$S_{34} = 17(142)$$

$$S_{34} = 2414$$

Section 2.3 – Practice Problems

EMERGING LEVEL QUESTIONS

Write the first four terms of each of the following sequences

1. $\{n^2 - 2\}$

2. $\left\{\frac{n+2}{n+1}\right\}$

3. $\{(-1)^{n+1}n^2\}$

4. $\left\{\frac{3^n}{2^n+1}\right\}$

5. $\left\{\frac{2^n}{n^2}\right\}$

6. $\left\{\left(\frac{2}{3}\right)^n\right\}$

Find the indicated arithmetic term

7. $a = 5, d = 3, find\ t_{12}$

8. $a = \frac{2}{3}, d = -\frac{1}{4}, find\ t_9$

9. $a = -\frac{3}{4}, d = \frac{1}{2}, find\ t_{10}$

10. $a = 2.5, d = -1.25, find\ t_{20}$

11. $a = -0.75, d = 0.05, find\ t_{40}$

12. $a = -\frac{7}{4}, d = -\frac{2}{3}, find\ t_{37}$

Find the number of terms in each arithmetic sequence

13. $a = 6, d = -3,\ t_n = -30$

14. $a = -3, d = 5,\ t_n = 82$

15. $a = 0.6, d = 0.2, t_n = 9.2$

16. $a = -0.3, d = -2.3,\ t_n = -39.4$

17. $-1, 4, 9, \ldots, 159$

18. $23, 20, 17, \ldots, -100$

PROFICIENT LEVEL QUESTIONS

Find the first term in the arithmetic sequence

19. 6th term is 10; 18th term is 46

20. 4th term is 2; 18th term is 30

21. 9th term is 23; 17th term is -1

22. 5th term is 3; 25th term is -57

23. 13th term is -3; 20th term is -17

24. 11th term is 37; 26th term is 32

Find the sum of the arithmetic series

25. $3 + 5 + 7 + \cdots + (2n + 1)$

26. $-1 + 2 + 5 + \cdots + (3n - 4)$

27. $2 + 5 + 8 + \cdots + 77$

28. $5 + 9 + 13 + \cdots + 97$

29. $(-41) + (-35) + (-29) + \cdots + 541$

30. $2\sqrt{5} + 6\sqrt{5} + 10\sqrt{5} + \cdots + 50\sqrt{5}$

31. $39 + 33 + 27 + \cdots + (-15)$

32. $23 + 19 + 15 +,,, + (-305)$

EXTENDING LEVEL QUESTIONS

Find the indicated value using the information given

33. $S_{20}, if\ a_1 = 8, a_{20} = 65$

34. $S_{21}, if\ a_1 = 8, a_{20} = 65$

35. $S_{56}, if\ a_{56} = 13, d = -9$

36. $n\ if\ S_n = 180, a_1 = 4, t_n = 16$

37. $d, if\ S_{40} = 680, a_1 = 11$

38. $S_{62}, if\ a_1 = 10, d = 3$

39. $S_{19}, if\ d = 4, a_{19} = 17$

40. $S_{40}, if\ d = -3, a_{40} = 65$

Section 2.3 – Answer Key

1. $-1, 2, 7, 14$
2. $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$
3. $1, -4, 9, -16$
4. $1, \frac{9}{5}, 3, \frac{81}{17}$
5. $2, 1, \frac{8}{9}, 1$
6. $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}$
7. $t_{12} = 38$
8. $t_9 = -\frac{4}{3}$
9. $t_{10} = \frac{15}{4}$
10. $t_{20} = -21.25$
11. $t_{40} = 1.2$
12. $t_{37} = -25.75$
13. $n = 13$
14. $n = 18$
15. $n = 44$
16. $n = 18$
17. $n = 33$
18. $n = 42$
19. $a = -5$
20. $a = -4$
21. $a = 47$
22. $a = 15$
23. $a = 21$
24. $a = 40\frac{1}{3}$
25. $n^2 + 2n$
26. $\frac{3n^2}{2} - \frac{5n}{2}$
27. $S_{26} = 1027$
28. $S_{24} = 1224$
29. $S_{98} = 24\,500$
30. $S_{13} = 338\sqrt{5}$
31. $S_{10} = 120$
32. $S_{83} = -11703$
33. $S_{20} = 730$
34. $S_{21} = 798$
35. $S_{56} = 14\,588$
36. $n = 18$
37. $d = \frac{4}{13}$
38. $S_{63} = 6293$
39. $S_{19} = -361$
40. $S_{40} = 4940$

Extra Work Space